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Graded non-commutative geometries

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We present a short exposition of graded finite non-commutative geometries. The theory that serves as an example is based on the algebra of matrices $M_n(\mathbb{C})$. This non-commutative algebra replaces the algebra of functions on a manifold. Consequently, vector fields (differentiations), forms and connections are constructed. The gauge theory can be introduced without the notion of internal manifold. We discuss some physical application, the similarities with the standard model, and the graded version of this geometry.

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1. Introduction

The stability of our physical world is apparently based on conservation laws. It has become clear, after Noether's theorems, that the existence of conserved quantities besides the energy, momentum and angular momentum, has its explanation in the existence of symmetry groups other than the Poincaré group. For example, the conservation of the electric charge can be derived from the invariance of the interactions under the abelian gauge group U(1), connected with the arbitrariness of the phase of a wave function. The fact that the group U(1) is compact leads to the *discrete* character of this particular conservation law, i.e. to the *quantization* of the electric charge. Other discrete conservation laws followed, the *baryon* number, the leptonic charge, isospin and strangeness – although not as universal as the electric charge conservation, except for the baryonic charge conservation, which seems to be *absolutely* satisfied. This situation has led to the introduction of the so-called internal symmetries, and it seemed quite natural to consider the appropriate "internal spaces" on which the compact Lie groups would act as the Poincaré group acts on the space–time manifold. The internal spaces should be compact and their characteristic dimensions very small in order to explain the orders of magnitude of the corresponding elementary interactions. By no means can they be penetrated by or mixed up with the "ordinary" space–time dimensions accessible to our experience. Such was the realm described by the Kaluza–Klein approach to the unified theories of elementary interactions and forces. The epistemological status of the internal spaces was therefore quite similar to that of the celestial spheres and epicycles of the Ptolemean (and even Copernican) astronomy. It took some time to acknowledge that one inverse square law of universal attraction introduced by Newton along with the equations of dynamics is enough to explain the observed motion of celestial bodies and to dispose of all the celestial spheres, untouchable and impenetrable by our senses.

It seems, upon closer inspection, that the same fate may be predicted for the "internal spaces" of the Kaluza–Klein theories. The useful information that can be drawn from the conservation laws implemented by the compact and semisimple Lie groups is always *discrete*; as a matter of fact, it all can be read from the root diagrams or Dynkin diagrams. If so, why bother about the non-observable internal manifolds containing the infinite quantity of points, with all possible C^∞-functions, the vector fields, and all the richness of differential geometry? A more economical approach would consist of liberating the mathematical background of the theory from all unnecessary artefacts, leaving only the minimal (and therefore *discrete*) input that is needed in order to get the apparently *discrete* output contained in the finite spectrum of charges and masses of the observed elementary particles and fields.

This is the motivation for the development of a different approach, in which the internal geometry would reduce to the very minimum, which is probably a discrete set of information. The attempts to construct such geometries go back to Quillen [1,2], and Connes and Lott [3]; they have been developed in a series of works by Dubois-Violette, Madore and Kerner [4–6] and their application to the construction of unified models has been further developed by Coquereaux and Esposito-Farese [7]. All these approaches have in common that they can be thought of as the internal geometries of a point or of a finite set of points rather than the geometries of manifolds. The full theory can be constructed starting with a non-commutative finite or denumerable differential algebra, which replaces the algebra of C^∞-functions in the usual case. The derivations of this algebra play the role of vector fields; their dual space is the space of one-forms, and so forth. Consequently, a simplified theory of integration, Hodge duality and de Rham complex, as well as the Laplace–Beltrami operator with a discrete spectrum can be introduced and incorporated into a simplified version of gauge theory.

The aim of this paper is to awake above all the curiosity of the reader and to convince him (her) that it is worthwhile to read the papers cited above. We do not intend to give a detailed exposition here; we shall rather announce the most important results and illustrate them with a few very simple examples. We shall mostly insist on the possibility of constructing a gauge theory without internal manifold, or more precisely, with a manifold reduced to a few points.

2. Non-commutative geometry without grading

One of the most elegant ways of defining a differentiable manifold V_n consists of considering the algebra of all smooth (C^k or C^∞) functions on this manifold, denoted by $F(V_n)$. The points of the manifold are then identified with the *maximal ideals* of this algebra, which coincide with the sets of all functions vanishing at a given point p: as a matter of fact, if $f(p)=0$, then $g(p)f(p)=0$ for any $g\in F(V_n)$.

The *vector fields* are identified with the *derivations* of the algebra $F(V_n)$; they form a *left module* over $F(V_n)$. Next, in the usual way, one constructs the dual module of *one-forms*, the tensorial products of these, and so on (cf. Kobayashi and Nomizu [8]). To the Cartesian products of two manifolds correspond the tensorial products of their algebras of functions; the connections can be put into a correspondence with modules over the algebra $F(V_n)$, these modules being the classes of local sections of given fibre bundles.

We shall show how easily all these notions can be transposed to the case when the algebra $F(V_n)$ is replaced by an associative, but *non-commutative* algebra. For the toy model we shall choose the algebra of $n\times n$ complex matrices, $M_n(\mathbb{C})$.

In that case, one can choose the canonical basis $E_k\in M_n(\mathbb{C})$, $k=1, 2, ..., n^2-1$ of hermitian traceless matrices, which together with the unit matrix 1 span the whole algebra. The matrices ("functions") E_k satisfy the following algebraic relations:

$$E_kE_l=\frac{1}{n}g_{kl}+S^m_{kl}E_m-\mathrm{i}C^m_{kl}E_m\,, \tag{1}$$

where C^m_{kl} are the structure constants of the Lie algebra $\mathrm{Sl}(n)$. One has then

$$[E_k, E_l]=-2\mathrm{i}C^m_{kl}E_m\,. \tag{2}$$

S^l_{mk} are symmetric and traceless invariant $\mathrm{Sl}(n)$ tensors; also

$$g_{kj}=\mathrm{Tr}(E_kE_j)\,. \tag{3}$$

The derivations of our algebra $M_n(\mathbb{C})$ span the vector fields, which can be denoted by ∂_k:

$$\partial_kE_j\stackrel{\mathrm{df}}{=}\mathrm{ad}(\mathrm{i}E_k)E_j=\mathrm{i}[E_k, E_j]=2C^m_{kj}E_m\,. \tag{4}$$

It is important to note that the derivations $\partial_k\in\mathrm{Der}(M_n(\mathbb{C}))$ do not form a module over the algebra $M_n(\mathbb{C})$, because $E_l\partial_k$ is not a derivation. The Leibniz rule

$$\partial_k(E_{jl}E_m)=(\partial_kE_j)E_m+E_j(\partial_kE_m) \tag{5}$$

is assured by the Jacobi identity. One has

$$[\partial_k, \partial_j]=2C^m_{kj}\partial_m\,, \tag{6}$$

and $\partial_k 1 = 0$ by definition. The dual basis of exterior one-forms can be now defined by

$$\theta^k(\partial_m) = \delta^k_m 1 . \tag{7}$$

The exterior derivation satisfying the graded Leibniz rule,

$$d(\theta^k \wedge \theta^j) = d\theta^k \wedge \theta^j - \theta^k \wedge d\theta^j \tag{8}$$

and

$$d^2 = 0 \tag{9}$$

can be easily defined if we put

$$dE_k = 2C^j_{km} E_j \theta^m , \quad d1 \equiv 0 , \tag{10}$$

$$d\theta^k = -C^k_{jm} \theta^j \wedge \theta^m . \tag{11}$$

Equation (11) is the non-commutative analogue of the Maurer–Cartan identity. It is easy to show that

$$\theta^k = E_m E^k \, dE^m \tag{12}$$

and that the canonical one-form θ,

$$\theta = E_k \theta^k , \tag{13}$$

is basis independent.

Denoting by i_x the anti-derivation of p-forms defined as

$$(i_x \alpha)(X_1, ..., X_{p-1}) = \alpha(X, X_1, ..., X_{p-1}) , \tag{14}$$

we can introduce the Lie derivative with respect to any vector field x as usual:

$$\pounds_x \alpha = (d \circ i_x + i_x \circ d) \alpha . \tag{15}$$

It is easy to show that

$$\pounds_x \theta = 0 \tag{16}$$

for any $X \in \mathrm{Der}(M_n(\mathbb{C}))$. Finally, the notion of a volume element,

$$\eta = \sqrt{|g|} \, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1} \tag{17}$$

and the star ($*$)-Hodge isomorphism can be defined, as well as the integration of (n^2-1)-forms; indeed, if β is a (n^2-1)-form, it must be equal to

$$\beta = B\sqrt{|g|} \, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1} \tag{18}$$

and one can then define

$$\int \beta = \frac{1}{n} \mathrm{Tr}(B) . \tag{19}$$

The scalar product of two p-forms is defined as usual:

$$\langle \alpha | \beta \rangle = \int \bar{\alpha} \wedge (*\beta) \, . \tag{20}$$

The anti-derivation δ is introduced by the identity

$$\langle d\alpha | \beta \rangle = \langle \alpha | \delta\beta \rangle \, , \tag{21}$$

which amounts to

$$\delta\alpha = (-1)^{(n^2-1)p+n^2} * d * \alpha \, , \tag{22}$$

if α is a p-form.

The Laplace–Beltrami operator is then

$$\Delta = d\delta + \delta d \tag{23}$$

and has a *finite discrete* spectrum on all possible forms and functions in our Grassmann algebra.

This is what can be called "the geometry of a point", which turns out to be surprisingly rich. This geometry should replace the geometry of the "internal space" in usual Kaluza–Klein theories. In order to do so, we must generalize the notion of connection. This is done as follows: the best way to define a connection in some bundle is to give the rule for *covariant derivation* of local sections of the bundle, which form naturally a *module* over the algebra of functions on the basis with values in the structural group acting on the fibres. If f is an element of this module, and A is the element of the algebra acting on f on the right, then a covariant derivative is a linear mapping satisfying the Leibniz rule:

$$\nabla(\phi A) = (\nabla\phi)A + \phi \otimes dA \, . \tag{24}$$

Let $\mathscr{H}$ be a hermitian module of rank 1 over our non-commutative algebra, which means that there is a scalar product defined for any two ϕ, $\psi \in \mathscr{H}$:

$$h(\phi, \psi) \in M_n(\mathbb{C}) \, . \tag{25}$$

We impose

$$d(h(\phi, \psi)) = h(\nabla\phi, \psi) + h(\phi, \nabla\psi)) \, . \tag{26}$$

Let $e \in \mathscr{H}$ be a unitary element, i.e.,

$$h(e, e) = 1 \, . \tag{27}$$

Now every element of the module $\mathscr{H}$ can be put in one-to-one correspondence with the elements of the algebra: for each $\phi \in H$, there exists $A \in \mathscr{A}$ such that

$$\phi = eA \, , \tag{28}$$

and we have

$$h(eA, eB) = A^*B\ , \quad A, B \in M_n(\mathbb{C})\ . \tag{29}$$

The choice of the element e is equivalent with the choice of a gauge in ordinary theory. A *gauge transformation* is just another choice of a *unitary* element, induced by right action of a unitary matrix $U \in M_n(\mathbb{C})$:

$$e \to e' = eU\ . \tag{30}$$

According to the definition of covariant derivation, if $\phi = eA$, then

$$\nabla\phi = (\nabla e)A + e \otimes dA\ , \tag{31}$$

$$\nabla e = e \otimes \alpha\ , \tag{32}$$

where the one-form $\alpha \in \wedge^1(M_n(\mathbb{C}))$ is unique if we require it to be the hermitian, i.e., to satisfy

$$\bar{\alpha} = \alpha\ . \tag{33}$$

The elements B and α are called *the components of ϕ and ∇ in the gauge e.* Under a change of gauge $e \to e' = eU$, $U \in \mathrm{U}(n)$, the components do transform as usual:

$$B \to U^{-1}B\ , \quad \alpha \to U^{-1}\alpha U + U^{-1}\alpha U\ , \tag{34}$$

which is the exact analogue of a gauge theory. There is one important difference, however, concerning the vacuum configurations and their correspondence with a pure gauge. Let us recall that, given a gauge e, there is a *unique* connection $\nabla^{(e)}$ such that

$$\nabla^{(e)}e = 0\ , \quad \text{i.e.,} \quad \nabla^{(e)}(eB) = e \otimes dB\ . \tag{35}$$

Under a change of gauge $e \to eU$ this connection has the form of a "pure gauge" connection:

$$\nabla^{(eU)} \to \alpha = U^{-1}dU\ . \tag{36}$$

Such connections have vanishing curvature:

$$\nabla^2 e = e \otimes \varphi\ , \quad \varphi \in \wedge^2(M_n(\mathbb{C}))\ , \tag{37}$$

and obviously, if $\nabla e = e \otimes \alpha$, then

$$\varphi = d\alpha + \alpha \wedge \alpha\ . \tag{38}$$

Under a gauge transformation the curvature two-form φ transforms uniformly:

$$\varphi \to U^{-1}\varphi U\ . \tag{39}$$

Obviously, the pure gauge connections have vanishing curvature: as $\nabla e = 0$, $\nabla(eB) = e \otimes dB$, so that

$$\nabla^2(eB) = \nabla(\nabla(eB)) = (\nabla e) \otimes dB + e \otimes ddB = 0 \tag{40}$$

because of $\nabla e=0$ and $ddB=0$.

But here there exists a connection whose curvature is vanishing but which is *not* a pure gauge. Consider the covariant derivation defined as follows:

$$\nabla\phi=-\mathrm{i}\phi\otimes\theta\,, \tag{41}$$

where $\theta=E_k\theta^k$ is the canonical one-form defined before, such that $d\theta+\theta\wedge\theta=0$.

It can be easily shown that this connection is gauge invariant, i.e., it cannot vanish in any gauge, but at the same time its curvature is identically vanishing. The proof is almost immediate if one notes that for any $B\in M_n(\mathbb{C})$ we can write

$$dB=\mathrm{i}[\theta,B]=\mathrm{i}\theta B-\mathrm{i}B\theta\,. \tag{42}$$

Then, because of

$$\nabla(\phi B)=-\mathrm{i}(\phi B)\theta=-\mathrm{i}\phi\theta B+\phi\mathrm{i}[\theta,B]=(\nabla\phi)B+\phi\,dB\,, \tag{43}$$

∇ is obviously a connection; under a change of gauge it transforms as

$$\begin{aligned}U^{-1}(-\mathrm{i}\theta)U+U^{-1}dU&=U^{-1}(-\mathrm{i}\theta)U+U^{-1}\mathrm{i}[\theta,U]\\&=U^{-1}(-\mathrm{i}\theta)U+U^{-1}\mathrm{i}(\theta U-U\theta)=-\mathrm{i}\theta\,,\end{aligned} \tag{44}$$

which proves that it cannot be gauged out.

Finally, the curvature ∇^2 is zero, because

$$d\phi+\phi\wedge\phi=0 \tag{45}$$

as noted before.

The existence of a stable vacuum configuration that is not a pure gauge is a new and important feature of the non-commutative gauge theory. Besides that, all other aspects of the theory can be transplanted without any difficulty, to give the final Lagrangian of the well-known form

$$\langle\varphi|\varphi\rangle=-\int\varphi\wedge{}^*\varphi=\tfrac{1}{2}\|\nabla^2\|^2\,. \tag{46}$$

In coordinates, if we choose to put

$$\alpha=\beta-\mathrm{i}\theta=B_k\theta^k-\mathrm{i}E_k\theta^k\,, \tag{47}$$

we obtain the expression

$$\|\nabla^2\|^2=-\frac{1}{8n}\sum_{k,j}\mathrm{Tr}\{([B_k,B_j]-2C^m_{kj}B_m)^2\}\,. \tag{48}$$

The two distinct gauge orbits mentioned above correspond to the *minima* of this action, which are either $B_k=0$, or $B_k=E_k$.

This "gauge theory of internal space" can now serve to implement the usual field theory on the space–time manifold V_4. The essential point comes from the observation that

$$\mathrm{Der}[\mathrm{C}^\infty(V_n)\otimes M_n(\mathbb{C})]=[\mathrm{Der}(\mathrm{C}^\infty(V)\otimes 1)] \\ +[\mathrm{C}^\infty(V_n)\otimes \mathrm{Der}(M_n(\mathbb{C}))]\ , \qquad (49)$$

which means that any derivation (a generalized vector field) of the algebra which is the tensor product of the usual algebra of functions over the manifold by $M_n(\mathbb{C})$ can be expressed locally as

$$X=X^\mu(x)\partial_\mu\otimes 1+\xi^k(x)\partial_k\ , \qquad (50)$$

with $x\in V_4$, $\mu=0, 1, 2, 3$, $l=1, 2, ..., n^2-1$, and X^m, $\xi^l\in \mathrm{C}^\infty(V_4)$. This means in turn that the connection one-form (A) can be written as

$$\alpha=A^0_\mu(x)\ dx^\mu+B^0_j(x)\theta^j+A^k_\mu(x)E_k\ dx^\mu+B^j_kE_j\theta^k\ . \qquad (51)$$

It unifies the $\mathrm{U}(1)\times\mathrm{SU}(n)$ gauge fields $A^0{}_\mu$ and $A^k{}_\mu$ with scalar multiplets (Higgs fields) $B^0{}_j$ and $B^m{}_k$. The curvature two-form splits into five distinct parts,

$$F=d\alpha+\alpha\wedge\alpha \\ =F^0_{\mu\nu}\ dx^\mu\wedge dx^\nu+F^0_{\mu k}\ dx^\mu\wedge\theta^k+G^j_{\mu\nu}E_j\ dx^\mu\wedge dx^\nu \\ +F^j_{km}E_j\theta^k\wedge\theta^m+F^j_{\mu m}E_j\ dx^\mu\wedge\theta^m\ . \qquad (52)$$

There is one dimensional parameter m needed to give the proper dimension (cm^{-1}) to the non-commutative one-forms θ^k; the full lagrangian of the theory is then

$$L=-\frac{1}{4n}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})-\frac{m^2}{2n}\mathrm{Tr}\{(\partial_\mu B_k+[A_\mu, B_k])^2\} \\ -\frac{m^4}{4n}\mathrm{Tr}\{([B_k, B_l]-2C^m_{kl}B_m)^2\}\ . \qquad (53)$$

On the gauge orbit $B_k=0$ the fields $A^0{}_\mu$ and $A^l{}_\mu$ are massless, whereas the fluctuations ϕ_k of the field B_k have all the same mass $M^2_\phi=nm^2$. If one expands the fields around another stable gauge orbit $B_k=E_k$, then the $\mathrm{U}(1)$ gauge field $A^0{}_\mu$ remains massless, whereas the $\mathrm{SU}(n)$ gauge field $B^l{}_\mu$ acquires the mass $M^2_B=2nm^2$; at the same time the scalar multiplet acquires the mass $M^2_B=2nm^2$ whereas $B^l{}_k$ splits into one scalar, one traceless symmetric and one anti-symmetric part, whose masses are, respectively, $2m^2$, $8m^2$ and zero. This is a very stiff kind of model, with only one arbitrary parameter m, with the same masses for the W and Z-bosons, and without mixing of $A^0{}_\mu$ and $B^3{}_\mu$. Nevertheless, it is a good example illustrating the possibility of constructing a gauge theory *without* internal manifold, which is replaced by a point with a non-commutative geometry on it.

3. Grading of the non-commutative geometry

A graded version of the non-commutative geometry can be introduced as follows. First, let us consider a Z_2-version, on the very simple example of the 2×2 matrix algebra. Such an algebra can be naturally split into a sum of two linear subspaces, $M_2(\mathbb{C}) = A_0 \otimes A_1$, with A_0 the "even" diagonal matrices, and A_1 the "odd" off-diagonal matrices. If Γ is a "grading matrix", $\Gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then, if we put

$$G^{-1}AG = \pm A\,, \tag{54}$$

the + and − signs define precisely our two subspaces. We shall attribute grade 0 to any $a_0 \in A_0$, and grade 1 to any $a_1 \in A_1$. Then

$$a_k b_m \in A_{(k+m) \bmod 2}\,. \tag{55}$$

The exterior *graded* differentiation can be now defined without vector fields or forms: it is enough to take an *odd* matrix whose square is equal to the identity, e.g.,

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta^2 = 1\,, \tag{56}$$

and define

$$da = [\eta, a] Z_2 = \eta a - (-1)\,\mathrm{grad}^{(a)} a\eta\,. \tag{57}$$

One has then $d^2 = 0$, as it should be.

Now we have gone even further than in the previous section: we have disposed not only of the manifold itself, but also of the vector fields (the tangent manifold) and forms (cotangent manifold). All the entities are dissolved in one *graded* algebra of matrices, on which the *exterior differential* is defined, the only thing, along with the scalar product, that is needed to produce a lagrangian. The scalar product will be defined as the graded trace, in conformity with our previous scheme.

Now we can produce an algebra of A-valued forms on the manifold V_4:

$$\wedge(V_4) \otimes A\,. \tag{58}$$

Then the operator d can be generalized as follows: If $A = \left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b \\ f & 0 \end{smallmatrix}\right)$, with a, b, c, f some p-forms over V_4, then

$$dA = \begin{pmatrix} dA & 0 \\ 0 & dc \end{pmatrix} + \begin{pmatrix} 0 & db \\ df & 0 \end{pmatrix} + \left[\eta, \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\right] + \left\{\eta, \begin{pmatrix} 0 & b \\ f & 0 \end{pmatrix}\right\}_+\,, \tag{59}$$

which means that its action is as usual on space–time p-forms, implemented with the "external" derivation realized as a graded commutator with the odd matrix η.

Now, a covariant derivation can be introduced exactly as before; the Higgs $M_2(\mathbb{C})$-valued field appears naturally if we introduce the connection one-form as

$$\omega = A + \chi\,, \tag{60}$$

with $\chi = \eta + \phi$, A an ordinary one-form with its values in $M_2(\mathbb{C})$; the simplest connection may be chosen as

$$A = \begin{pmatrix} A^0 & W^+ \\ W^- & Z \end{pmatrix}, \tag{61}$$

with A^0, W^+, W^-, Z one-forms over V_4.

This scheme is not enough to explain the mass difference between W and Z bosons. The mass difference appears in the following generalization: we can introduce a $U_2 \times U_2$ multiplet, in which each of the entries, A^0, W^+, W^-, Z is a 2×2 complex matrix, and the connection one-form becomes

$$A = A_+ \otimes P_+ + A_- \otimes P_-\,, \tag{62}$$

with P_+ and P_- the projectors

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{63}$$

and

$$A_\pm = \begin{pmatrix} A^0_\pm & W^+_\pm \\ W^-_\pm & Z_\pm \end{pmatrix}, \quad A_+ A_- = 0\,. \tag{64}$$

$A_\pm$ are the U_2 gauge bosons that couple to left- and right-handed fermions, respectively. The connection $A_\pm$ contains 32 real, left- and right-handed U_4 modes. To reduce the symmetry, we choose the vacuum configuration around the gauge orbit given by

$$\phi = \mu_a \tau^a\,, \quad \mu_a = m P_- \otimes \sigma_a\,, \tag{65}$$

where m is the mass parameter, $a = 1, 2, 3$, σ_a are the 2×2 Pauli matrices.

Developing the lagrangian

$$L = \| \nabla A \|^2 \tag{66}$$

around this vacuum, we obtain a massless field A^0, whereas $m_W^2 = \frac{3}{2} m^2$, $m_Z^2 = 2m^2$, so that

$$m_Z^2 / m_W^2 = 0.75\,, \tag{67}$$

which is reasonably close to the observed value and predicts the Weinberg angle $\theta_W = 30°$.

To close this section, let us mention another graded extension of matrix geometry, based on the Z_3 grading. It leads to a differential operator d whose square does not vanish, but whose cube d^3 is identically zero. The algebra of 3×3 matrices can be naturally split into three parts, with grades 0, 1 and 2, adding up modulo 3; instead of the two eigenvalues of the grading matrix, we have now *three* eigenvalues

$$1, j, j^2, \quad j = e^{2\pi i/3}, \quad j + j^2 + j^3 = 0 . \tag{68}$$

The graded derivation satisfies

$$d(AB) = (dA)B + j^{\mathrm{grad}(A)} A\, dB , \tag{69}$$

which assures $d^3 = 0$. The covariant derivation can also be introduced, but the curvature is not a covariant quantity; instead, we have to take

$$\nabla^3 = (d+A)(d+A)(d+A) = d^2A + d(A^2) + AdA + A^3 , \tag{70}$$

which contains third-order powers in A as well as the terms of the form $dA \wedge A$. The details can be found in refs. [9] and [10].

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